A Riemannian Framework for Ensemble Average Propagator Computing

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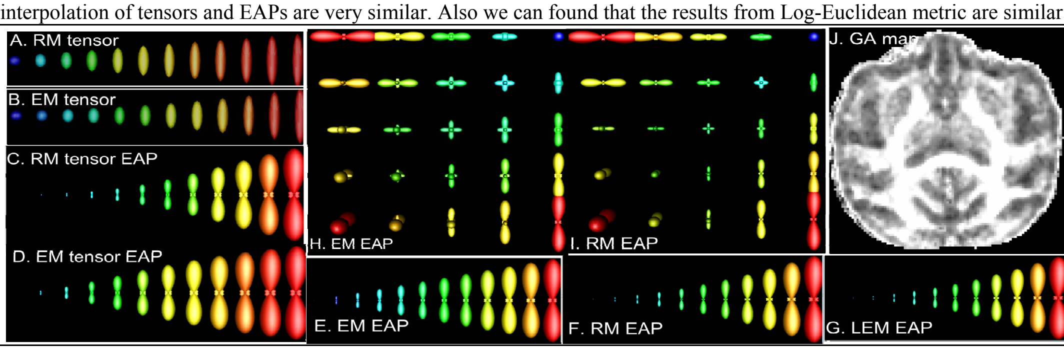
Introduction. In Diffusion Tensor Imaging (DTI), Riemannian framework (RF) [1] has been proposed for processing tensors, which is based on Information Geometry theory. Many papers have shown that RF is useful in tensor estimation, interpolation, smoothing, regularization, segmentation and so on. Recently RF also has been proposed for Orientation Distribution Function (ODF) computing [2,3] and it is applicable to any Probability Density Function (PDF) based on any orthornormal basis representation. Spherical Polar Fourier Imaging (SPFI) [4,5] was proposed recently to fast and robustly estimate the ODF and Ensemble Average Propagator (EAP) from arbitrary sampled DWI signals. In this paper, we propose the RF for EAPs and implement it via SPFI. We proved that the RF for EAPs is diffeomorphism invariant, which is the natural extension of affine invariant RF for tensors. It could avoid the so-called *swelling effect* for interpolating EAPs, just like the RF for tensors. We also propose the Log-Euclidean framework (LEF), Affine-Euclidean framework (AEF), for fast processing EAPs, and Geometric Anisotropy (GA) for measuring the anisotropy of EAPs, which are all the extensions of previous concepts in RM for tensors respectively.

Theory. In the seminal work on RF for ODFs [2], the ODF is represented by an orthonormal basis, $\varphi(\boldsymbol{u})=(\sum_i c_i B_i(\boldsymbol{u}))^2$, where normally Spherical Hamonic (SH) basis $\{Y_l^m(\boldsymbol{u})\}$ is chosen. Then the Riemannian Coordinate $\{c_i\}$ could be used for many operations (smoothing, interpolation etc.). RF could be constructed in any PDF under an orthonormal basis representation. We represent the EAP as $P(\boldsymbol{R})=(\sum_i c_i B_i(\boldsymbol{R}))^2$. In quantum mechanics, the square root of the probability of finding the subject at a certain time and position is called as wave function. Analogously, the square root of EAP, $\psi(\boldsymbol{R})=\sum_i c_i B_i(\boldsymbol{R})$, is also called as wave function. It could be proved easily that the Fisher information metric is diffeomorphism invariant. For any diffeomorphism $g: R^3 \to R^3$ in r-space and for any EAP $P_1(\boldsymbol{R})$ and $P_2(\boldsymbol{R})$, the Riemannian distance $dist(P_1(\boldsymbol{R}),P_2(\boldsymbol{R}))$ is equal to $dist(P_1(g(\boldsymbol{R})),P_2(g(\boldsymbol{R})))$. And if we fix $P(\boldsymbol{R})$ as a Gaussian distribution, then g will be an affine transform, which means that the RF for tensors is actually the limitation case of our RF for EAPs. This general property is probably much useful in EAP registration. Based on the orthonormal basis representation of the wave function $\psi(\boldsymbol{R})$, many theoretical results in [2,3] could be generalized into EAPs. The weighted Riemannian mean [2] and Riemannian median [3] both uniquely exist in this framework. Moreover, LEF and AEF for EAPs could be used for approximation of RF by projecting the EAPs into the tangent space of the isotropic EAP. The isotropic ODF is unique. However, the isotropic EAP is not unique. In LEF for tensors, the isotropic tensor is not unique and the identity tensor was chosen and fixed. Analogously we also just choose a typical Gaussian distribution and fixed for all EAPs.

Implementation. Theoretically speaking, the RF for EAPs could be implemented by any orthonormal basis. However, so far there is no direct way to estimate the Riemannian Coordinate $\{c_i\}$ from DWI signals. Existing reconstruction methods only estimate the EAP, and unfortunately most of them have much estimation error and unrealistic assumptions which cannot be used to represent all EAPs. SPFI is a fast, regularized, robust method to estimate EAPs, which does not need any assumption on EAPs. In SPFI [4,5], the signal in q-space is represented by an orthonormal basis, named as SPF basis, $E(\boldsymbol{q})=\sum_{n,l,m} a_{n,l,m} R_n(q) Y_l^m(\boldsymbol{u})$, where $R_n(q)$ is the n order Gaussian-Laguerre polynomial. Cheng et al [5] proved that the EAP could be represented analytically as $P(\boldsymbol{R})=\sum_{n,l,m} a_{n,l,m} D_{n,l}(R) Y_l^m(\boldsymbol{r})$ with the same coefficients $\{a_{n,l,m}\}$ where $D_{n,l}(R)$ is the *l* order spherical Hankel transform of $R_n(q)$ and $\{D_{n,l}(R)Y_l^m(\boldsymbol{r})\}$ is another family of function for EAPs in r-space. Based on the Parseval's theorem, $\{D_{n,l}(R)Y_l^m(\boldsymbol{r})\}$ actually form an orthonormal basis in r-space, which we call the Fourier dual Spherical Polar Fourier (dSPF) basis. So after estimating the coefficients $\{a_{n,l,m}\}$, the EAP is already under a linear representation of orthonormal basis $\{D_{n,l}(R)Y_l^m(\boldsymbol{r})\}$. To estimate the Riemnnian Coordinates $\{c_i\}$, the idea is to discretize the r-space to get many samples of EAP and then get the square root of the samples. Then the $\{c_i\}$ is estimated by a least square fitting.

Application: Geometric Anisotropy (GA). We propose the GA for EAPs. It is defined as the Riemannian distance from the EAP to the nearest isotropic EAP, which is a generalized version of GA for tensors. It could be proved easily that the nearest isotropic EAP for a given EAP $\{c_i\}$ is just the normalized version of the isotropic part of $\{c_i\}$. Fig. (J) shows a GA map for a real monkey data, we estimate the EAP using SPFI from 3 shell data (b=500,1500,3000s/mm^2) and then calculate the GA based on the estimated EAP. It could be seen that the higher anisotropic region has higher GA.

Application: Interpolation. Like the RF for tensors and ODFs, the RF for EAP could be used for interpolation. We demonstrate the Lagrange interpolation of tensors and EAPs in 1D and 2D, and compare the Riemannian metric with the Euclidean metric. In Fig. 2, we first generate the DWI signals from two fixed DTI tensors with eigenvalues [1.7,0.3,0.3]*1e-3mm^2/s and [0.3,0.3,0.3]*1e-3 mm^2/s. The EAPs are estimated using SPFI from the DWI signals using order 6 for spherical part and order 3 for radial part. Then ten EAPs were interpolated between them using Riemannian metric, Log-Euclidean metric and Euclidean metric respectively. Another way to interpolate these two Gaussian EAPs is to first interpolate 10 tensors between these two tensors with Riemannian metric and Euclidean metric, and then get the EAP samples from the interpolated tensors via the analytical Fourier Transform of Gaussian distribution. In 2D case, EAPs were interpolated from 4 EAPs. We visualize the interpolated tensors and all EAP profiles at 15um in two ways. The glyphs were colored by GA values. The so called *swelling effect* could be found when interpolating tensors and EAPs with Euclidean metric. The Riemannian interpolation for tensors and for EAPs will avoid the swelling effect. The results from Riemannian interpolation of tensors and EAPs are very similar. Also we can found that the results from Log-Euclidean metric are similar with the Riemannian metric.



(A): Riemannian metric for tensors;
(B): Euclidean metric for tensors;
(C): EAPs from tensors in (a);
(D): EAPs from tensors in (b);
(E): Euclidean metric for EAPs;
(F): Riemannian metric for EAPs;
(G): Log-Euclidean metric for EAPs;
(H): Euclidean interpolation for EAPs in 2D;
(I): Riemannian interpolation for EAPs in 2D;
(J) GA map for monkey data.

Conclusion. RF has been successfully used for tensor calculation in [1] and recently was generalized to ODF computation in [2]. We generalize the previous RF to EAP computing. We showed that the RF for EAPs is diffeomorphism invariant, which is a natural extension of RF for tensors. LEF and AEF were proposed to approximate RF with linear computation. The interpolation of EAPs using Riemannian metric could avoid the so called *swelling effect*, just like interpolation of tensors. We also propose the Geometric Anisotropy, which is the generalized version of GA for tensors.

References [1] X. Pennec, et al, IJCV 2006. [2] J. Cheng et al, MICCAI 2009, [3] J. Cheng et al, ISMRM 2010, [4] J. Cheng et al, MICCAI 2010, [5] J. Cheng et al, MICCAI 2010.